Math 211 - Bonus Exercise 9 (please discuss on Forum)

1) Prove that there are no simple groups of order 308 and 364.

2) Using the Sylow theorems and Exercise 6 on this week's sheet to prove the following generalization of Lemma 14 in the notes:

Consider a subgroup $H \leq G$ of a finite group G, ***not necessarily normal****. Then for any prime p,* ***there exists*** *a Sylow p-subgroup of G whose intersection with H will be a Sylow p-subgroup of H.*

3) Describe the Sylow p-subgroups of the dihedral group for odd p.

4) Find a Sylow p-subgroup of the group $GL_2(\mathbb{F}_p)$ of invertible 2×2 matrices with entries in the field $\mathbb{F}_p$, with respect to matrix multiplication. We're not studying fields in this class, but you encountered this notion in Algebraic Structures: all this means is that the group $GL_2(\mathbb{F}_p)$ consists of invertible 2×2 matrices whose coefficients are taken to be residues mod p.

5) **Definition:** Let G be a group and p be a prime integer. Consider the action φ of the cyclic group of order p on $G^p := G \times \ldots \times G$, where the latter product contains p copies of G, that is given by:

$$\varphi : \mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}(G^p)$$

$$\varphi([1]) = T$$

$$T(g_1, g_2, \ldots, g_p) = (g_p, g_1, \ldots, g_{p-1})$$

The wreath product $G \wr \mathbb{Z}/p\mathbb{Z}$ is defined as the semidirect product $G^p \rtimes_\varphi \mathbb{Z}/p\mathbb{Z}$. More generally, for all subgroups $H \leq S_n$ of the symmetric group and for all groups G, the wreath product $G \wr H$ is the semi-direct product $G^n \rtimes H$, where H acts on G^n by permutation of the components.

Prove that $\mathbb{Z}/p\mathbb{Z} \wr \ldots \wr \mathbb{Z}/p\mathbb{Z}$, where the latter product contains r copies of $\mathbb{Z}/p\mathbb{Z}$ as a Sylow p-subgroup of S_{p^r}. You can use the fact that there exists an injective homomorphism of groups $S_n \wr S_m \to S_{nm}$.